Structure and Classification of C*-algebras

Stuart White and Joachim Zacharias

Notes by

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Contents

Introduction

The following lecture notes were prepared during the *Master Class: Noncommutative geometry and quantum groups* which was held in Bedlewo (4.09 - 10.09.2016) and Warsaw (10.09 - 17.09.2016) as an opening school for the Banach Center *Simons Semester Noncommutative geometry the next generation*. The first part of this notes (sections 1-7) is based on the lectures given by Stuart White, while the second one (sections 8-11) covers the material presented by Joachim Zacharias. Together they provide an exposition of a recent concepts and developments concerning the subject of classification and structure of simple, separable and nuclear C*-algebras.

The author of this notes wish to thank Stuart White and Joachim Zacharias for a series of inspiring lectures and clarifying remarks. He also would like to thank Paul F. Baum, Alan Carey, Piotr M. Hajac and Tomasz Maszczyk for organizing the *Simons Semester Noncommutative geometry the next generation*, as well as express his gratitude to Adam Skalski for many fruitful discussions.

1 Preliminaries on von Neumann factors

In this section we present some basics facts concerning special class of von Neumann algebras. Recall that a von Neumann algebra M is a strongly (weakly) closed *-subalgebra of $B(H)$ which contains a unit. We say that a von Neumann algebra M is separable acting if $M \subset B(H)$ with H being a separable Hilbert space.

Definition 1.1. A factor is a von Neumann algebra M with trivial center, i.e. $Z(M) = \mathbb{C}\mathbb{1}$.

The following remarks show the importance of factors as a basic ingredients in the theory of von Neumann algebras.

Remark 1.2. Separable acting von Neumann algebra is a direct integral of factors.

Remark 1.3. A two-sided (strongly closed) ideal I in M is of the form pM where $p \in Z(M)$.

We say that projections $p, q \in M$ in von Neumann algebra are equivalent or Murray-von Neumann equivalent ($p \sim q$) if there exists $v \in M$ such that $p = v^*v$ and $q = vv^*$ and we say that p is subequivalent to q ($p \lesssim q$) if $p \sim q_0 \leqslant q$ (i.e. p is equivalent to some subprojection of q). Projection p is infinite if $p \sim p_0 \lneqq p$ (i.e. p is equivalent to its nontrivial subprojection) and p is finite if it is not infinite. Accordingly we say that a von Neumann algebra M is infinite if $\mathbb{1} \in M$ is infinite and M is finite if $\mathbb{1} \in M$ is finite.

Proposition 1.4. *In a factor any two projections are compatible, i.e.* $p \lesssim q$ *or* $q \lesssim p$*.*

sketch of a proof. Consider the largest subprojection $p_0 \leqslant p$ such that $p_0 \sim q_0 \leqslant q$. If $p = p_0$ or $q = q_0$ then the proof is completed. Suppose that it is not the case. Let $p_1 = p - p_0$ and $q_1 = q - q_0$.

By $U(M)$ we denote a set of unitary elements in M. Observe that the join $y = \bigvee_{u \in U(M)} up_1u^*$ is central, hence equal to identity (M is a factor). Indeed, for any $w \in U(M)$ we have

$$wyw^* = w\left(\bigvee_{u \in U(M)} up_1u^*\right)w^* = \bigvee_{u \in U(M)} wup_1(wu)^* = y,$$

therefore y commutes with all unitaries and what follows with each element in M (since any element in M is a linear combination of unitaries). If so then there exists $v \in U(M)$ such that $vp_1v^*q_1 \neq 0$. The partial isometry ω in a polar decomposition of $vp_1v^*q_1$ provides projections $p_2 \leqslant p_1$, $q_2 \leqslant q_1$ such that $p_2 \sim q_2$ (since $\omega\omega^* \leqslant vp_1v^*$ and $\omega^*\omega \leqslant q_1$). But by maximality p_1 and q_1 have no non-zero subprojections that are equivalent - contradiction. $\square$

Proposition 1.5 (Type decomposition; Murray, von Neumann)**.** *Let M be a separable acting von Neumann factor. The possible partial orders in $P(M)/\sim$ (where $P(M)$ is a set of projections in M) are given by*

- $\{0, 1, \ldots, n\}$ - *type* I_n *and* $M \cong M_n$,
- $\{0, 1, \ldots, \infty\}$ - *type* I_∞ *and* $M \cong B(\ell^2)$,
- $[0, 1]$ - *type* II_1,
- $[0, \infty]$ - *type* II_∞ *and* $M \cong \mathrm{II}_1 \overline{\otimes} B(\ell^2)$,
- $\{0, \infty\}$ - *type* III *(purely infinite).*

Example 1.6. Let Γ be a nontrivial discrete (countable) group. By $L\Gamma$ we denote the group von Neumann algebra, i.e. a von Neumann algebra generated by a left-regular representation

$$\lambda_g : \delta_h \mapsto \delta_{gh}$$

or in other words the smallest strongly closed subalgebra of $B(\ell^2(\Gamma))$ containing $\{\lambda_g : g \in \Gamma\}$. It can be shown that $L\Gamma$ is a factor of type II_1 if and only if Γ is ICC (Γ fulfills infinite conjugacy classes condition). This occur precisely when $\left|\{hgh^{-1} : h \in \Gamma\}\right| = \infty$ for all $g \neq e \in \Gamma$. Form that we have the following concrete examples of group von Neumann algebras which are II_1 factors (since the following group are ICC):

- $S_\infty = \bigcup_n S_n$, the group of permutation fixing all but finitely many integers,
- $\left\{ M = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{Q}, a \neq 0 \right\}$,
- $\mathbb{F}_n$, the free group of n generators.

Example 1.7. Let X be a standard probability space and let Γ acts on X by an action α in a free, ergodic and probability measure preserving way. Then $L^\infty(X) \rtimes_\alpha \Gamma$ is a II_1 factor.

Definition 1.8. A trace on a C*-algebra A (on a von Neumann algebra M) is a state $\tau : A \to \mathbb{C}$ ($\tau : M \to \mathbb{C}$) such that $\tau(ab) = \tau(ba)$ for all $a, b \in A$ ($a, b \in M$).

Fact 1.9. II_1 *factor has the unique trace (dimension of a projection, i.e.* $\tau : P(M) \to [0, 1]$*).*

Fact 1.10. *If* M *is a* II_1 *factor then* $p \lesssim q$ *if and only if* $\tau(p) \leqslant \tau(q)$*.*

Fact 1.11. *If* M *is finite (not necessarily a factor) then* $p \lesssim q$ *if and only if* $\tau(p) \leqslant \tau(q)$ *for all traces (for all* $\tau \in T(M)$*).*

Definition 1.12. Let M be a von Neumann algebra represented on $B(H)$ ($M \subseteq B(H)$). M is injective if there exists some linear map Φ such that $\|\Phi\| \leqslant 1$ and the following diagram

$$\begin{array}{ccc} B(H) & \xrightarrow{\ \Phi\ } & M \\ \cup\!\shortmid & \nearrow_{\mathrm{id}_M} & \\ M & & \end{array}$$

commutes. Then Φ is unital completely positive map (ucp map).

Remark 1.13. Injectivity is a property of M. It does not depend on the embedding of M into $B(H)$.

Proposition 1.14. $L\Gamma$ *is injective if and only if* Γ *is amenable. Let* α *be a free, ergodic and probability measure preserving action of* Γ *on* X*, then* $L^\infty(X) \rtimes_\alpha \Gamma$ *is injective if and only if* Γ *is amenable.*

Theorem 1.15 (Connes, 1974)**.** *There exists the unique injective* II_1 *factor.*

Remark 1.16. The important part of Connes's proof is to show that "abstract structure" (injectivity) implies "internal local approximation" (hyperfiniteness).

Definition 1.17. Separable acting von Neumann algebra M is hyperfinite if there exists a family of finite dimensional C*-algebras $F_1 \subseteq F_2 \subseteq \ldots \subseteq M$ such that $\bigcup_n F_n$ is dense (strongly) in M.

Example 1.18. $LS_\infty = \overline{\bigcup_n LS_n}$ is hyperfinite.

Proposition 1.19 (Murray, von Neumann)**.** *There exists the unique hyperfinite* II_1 *factor.*

2 Semidiscreteness, nuclearity, and nuclear dimension

The goal of this section is to establish relation between von Neumann algebras (notion of semidiscreteness) and C*-algebras (notion of nuclearity) in order to provide new tools for problems related to the classification of C*-algebras. In particular we give the definition of so called nuclear dimension of a C*-algebra. The importance of the aforementioned notion is justified by the following theorem.

Theorem 2.1. *The class of unital separable simple C*-algebras of finite nuclear dimension is classified by the Elliott invariant ("K-theory and traces") in the presence of the UCT (Universal Coefficient Theorem).*

UCT should be treated here as some sort of technical assumption on the level of KK-theory.

Definition 2.2. A von Neumann algebra M is semidiscrete if there exists a finite dimensional approximation, i.e. there is a net $\{(\psi_i, \phi_i, F_i)\}_i$ consisting of completely positive contractive maps (cpc maps) ϕ_i, ψ_i and finite dimensional C*-algebras F_i such that the following diagram

$$\begin{array}{ccc} M & \xrightarrow{\mathrm{id}_M} & M \\ & \searrow^{\psi_i} & \uparrow{\scriptstyle \phi_i} \\ & & F_i \end{array}$$

approximately commutes in the sense that $\phi_i\psi_i(x) \to x$ in weak* topology for any $x \in M$ (pointwise convergence with respect to the weak* topology topology).

Remark 2.3. First step of Connes's proof (cf. theorem 1.15) is establishing the fact that injectivity implies semidiscreteness.

Remark 2.4. When we know that M is hyperfinite, we can arrange for ϕ_i to be *-homomorphisms.

Definition 2.5. A C*-algebra A is nuclear (has a completely positive approximation) if there is a net $\{(\psi_i, \phi_i, F_i)\}_i$ consisting of completely positive contractive maps (cpc maps) ϕ_i, ψ_i and finite dimensional C*-algebras F_i such that the following diagram

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}_A} & A \\ & \searrow^{\psi_i} & \uparrow{\scriptstyle \phi_i} \\ & & F_i \end{array}$$

approximately commutes in the sense that $\|\phi_i\psi_i(x) - x\| \to 0$ for any $x \in A$ (pointwise convergence with respect to the norm topology).

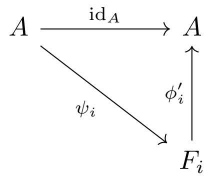
Proposition 2.6. *A C*-algebra A is nuclear if and only if A^{**} is semidiscrete.*

sketch of a proof. $\Rightarrow$ Nontrivial, factor through Connes's result, no direct approach know to this day.

$\Leftarrow$ If A^{**} is semidiscrete then it has a finite dimensional approximation given by the diagram

$$\begin{array}{ccc} A & \overset{i}{\hookrightarrow} & A^{**} \\ & \searrow^{\psi_i} & \uparrow{\scriptstyle \phi_i} \\ & & F_i \end{array}$$

with $\phi_i\psi_i(x) \to x$ in weak* topology for any $x \in A \subseteq A^{**}$. By Kaplansky density theorem we may replace ϕ_i by maps $\phi_i' : F_i \to A$ (this is possible since there is a bijective correspondence between completely positive contractive maps form M_n to A^{**} and positive elements in $M_n(A^{**})$). As a result we obtain the following diagram



with $\phi_i\psi_i(x) \to x$ in weak-topology for any $x \in A$. By Hahn-Banach theorem point-weak closure of a convex sets coincides with its point-norm closure. Since convex combinations of factorable maps are factorable, therefore we get the desired norm convergence. ☐

Definition 2.7. A completely positive map $\Theta : B \to C$ is called order zero if it preserves orthogonality i.e. for any $x, y \in B_+$, $xy = 0$ implies $\Theta(x)\Theta(y) = 0$.

Example 2.8. Any *-homomorphism is an order zero map.

General form of an order zero map is given by

$$\Theta(x) = h^{\frac{1}{2}}\pi(x)h^{\frac{1}{2}}$$

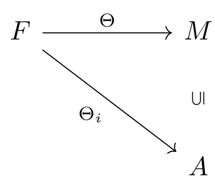
where h is positive, π is *-homomorphism $\pi : B \to \mathcal{M}(C^*(\Theta(B)))$ to multiplier algebra and $[h, \pi(x)] = 0$ for all $x \in B$ (cf. theorem 8.1).

Remark 2.9. Any unital order zero map is automatically a *-homomorphism.

Remark 2.10. There is a duality between order zero maps from B to C and cones over *-homomorphisms $C_0(0,1] \otimes B \to C$.

In that context there is an analogue of the Kaplansky density theorem.

Proposition 2.11. *Let F be a finite dimensional C^*-algebra with an order zero map $\Theta : F \to M$ to von Neumann algebra M. By A denote a strongly dense C^*-subalgebra in M. There exists a net $\{\Theta_i\}_i$ of order zero maps such that*



approximately commutes, i.e. $\Theta_i(x) \to \Theta(x)$ in strong(weak*) topology for any $x \in F$.*

By this proposition on can present a modified proof of implication $\Leftarrow$ from proposition 2.6, if A^{**} is hyperfinite. In that case one can replace *-homomorphisms by order zero maps in order to obtain approximately commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}_A} & A \\ & \searrow^{\psi_i} & \uparrow{\scriptstyle \phi_i'} \\ & & F_i \end{array} .$$

Proposition 2.12. *Let A be a nuclear C*-algebra. Then for any finite subset $\mathcal{F} \Subset A$ and any $\epsilon > 0$ there is a completely positive contractive (cpc) approximation, i.e. there are completely positive contractive maps ψ and ϕ such that*

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}_A} & A \\ & \searrow^{\psi} & \uparrow{\scriptstyle \phi} \\ & & F = F^{(0)} \oplus \ldots \oplus F^{(n)} \end{array}$$

where each $F^{(i)}$ is a finite dimensional C-algebra, $\|\phi\psi(x) - x\| < \epsilon$ for any $x \in \mathcal{F}$ and $\phi|_{F^{(i)}}$ is order zero map for all $i = 0, 1, \ldots, n$.*

Remark 2.13. Decomposition $F = F^{(0)} \oplus \ldots \oplus F^{(n)}$ depends on $\mathcal{F}$ and ϵ.

Definition 2.14. A C*-algebra A has nuclear dimension $\dim_{\mathrm{nuc}}(A) \leqslant n$ if for any finite subset $\mathcal{F} \Subset A$ and any $\epsilon > 0$ there is a completely positive (cp) approximation, i.e. there are completely positive maps ψ and ϕ such that

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}_A} & A \\ & \searrow^{\psi} & \uparrow{\scriptstyle \phi} \\ & & F = F^{(0)} \oplus \ldots \oplus F^{(n)} \end{array}$$

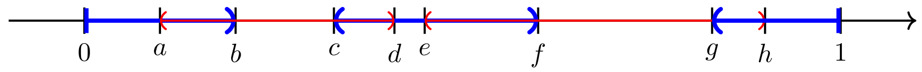
where each $F^{(i)}$ is a finite dimensional C*-algebra, $\|\phi\psi(x) - x\| < \epsilon$ for any $x \in \mathcal{F}$, $\phi|_{F^{(i)}}$ is contractive order zero map for all $i = 0, 1, \ldots, n$ and $\|\psi\| \leqslant 1$.

Remark 2.15. Decomposition $F = F^{(0)} \oplus \ldots \oplus F^{(n)}$ depends on $\mathcal{F}$ and ϵ.

Remark 2.16. If ϕ in the above definition is contractive then A is quasidiagonal.

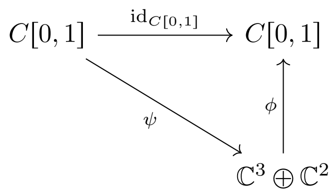
Example 2.17. $\dim_{\mathrm{nuc}}(C[0,1]) \leqslant 1$

Proof. Fix finite subset $\mathcal{F} \subset C[0,1]$ and $\epsilon > 0$. Find an open cover of $[0,1]$ such that elements in $\mathcal{F}$ are ϵ-constant. Without the loss of generality we can consider for example the open cover of the unit interval consisting of the following sets: $H_1 = [0,b)$, $H_2 = (c,f)$, $H_3 = (g,1]$, $K_1 = (a,d)$ and $K_2 = (e,h)$ where $0 < a < b < c < d < e < f < g < h < 1$ (we can always refine the given cover in order to obtain the generic cover such as presented here - two colorable refinement, see the figure below, when H_i are represented by color blue and K_i are represented by color red).



Let us choose a point x_i in each H_i and y_i in each K_i such that any of those points do not belongs to the intersection of any open sets in the described cover. Let h_i and k_i denote a partition of identity subordinant to the given cover (such that h_i is supported on H_i and k_i is supported on K_i), i.e. $\mathbb{1} = \sum_i h_i + \sum_i k_i$.

Define the following diagram



such that $\psi : C[0,1] \to \mathbb{C}^3 \oplus \mathbb{C}^2$ is given by evaluation maps

$$\psi = \mathrm{ev}(x_1, x_2, x_3) \oplus \mathrm{ev}(y_1, y_2)$$

and $\phi : \mathbb{C}^3 \oplus \mathbb{C}^2 \to C[0,1]$ is described by partition of unity

$$\psi(\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2) = \sum_i \lambda_i h_i + \sum_i \mu_i k_i.$$

For $f \in \mathcal{F}$ and typical $t \in [0,1]$ we have for example

$$f(t) = f(t)h_2(t) + f(t)k_2(t)$$

and

$$\phi\psi(f)(t) = f(x_2)h_2(t) + f(y_2)k_2(t).$$

Therefore, since f is ϵ-constant, we have $\|\phi\psi(f) - f\| < \epsilon$ for any $f \in \mathcal{F}$. Moreover, $\phi|_{\mathbb{C}^3}$ and $\phi|_{\mathbb{C}^2}$ are order zero contractive maps. $\square$

Remark 2.18. In fact it is an example of a more general phenomenon (cf. definition 8.5 and example 8.7).

There is a natural question when $\dim_{\mathrm{nuc}}(A) = 0$? Obviously it is true for finite dimensional C*-algebras. What is more, such a property is preserved by direct limits so in particular AF algebras has zero nuclear dimension. In fact $\dim_{\mathrm{nuc}}(A) = 0$ if and only if A is AF algebra (cf. section 8).

We will return to the notion of nuclear dimension in section 8. At the end of this part of notes let us consider the following table which implements nuclear dimension in the general scheme of defying analogous (corresponding) notions in the C*-algebras setting based on well know notions form von Neuman algebras theory.

von Neumann notions	direct C*-notions	colored C*-notions
hyperfiniteness	AF algebras	finite nuclear dimension
Rokhlin theorems	Rokhlin property	Rokhlin dimension
$M\overline{\otimes}R \cong M$	$A \otimes UHF \cong A$	$A \otimes Z \cong A$

Where R, UHF, Z are hyperfinite II_1 factor, universal UHF algebra and Jiang-Su algebra respectively.

3 Intertwining arguments

In this section we look closely at so called intertwining techniques and provide examples of how such methods from von Neumann algebras theory can be translated into C*-algebras setting. Intertwining techniques become extremely useful in a classification tasks, since in the presence of "internal approximation" (hyperfiniteness, being direct limit, etc.) classification of C*-algebra comes down to proving uniqueness theorems.

Example 3.1. Let F be finite dimensional C*-algebra and M denotes II_1-factor. Then linear maps

$$\varphi : F \to M$$

are classified by traces, i.e. φ_1 and φ_2 are unitarily equivalent if and only if $\tau \circ \varphi_1 = \tau \circ \varphi_2$ where τ is the unique trace on M. Given any trace tr on F there exists a map $\varphi : F \to M$ such that $\mathrm{tr} = \tau \circ \varphi$.

We wish to have an approximate version of this example with respect to the norm defined by $\|x\|_2 = \tau(x^*x)^{\frac{1}{2}}$ (this is an appropriate choice as this norm is metrizable with respect to strong* topology on the unit ball in M).

Fact 3.2. *For any finite dimensional C*-algebra F and $\epsilon > 0$, there exists $\delta > 0$ such that any two completely positive contractive maps $\phi_1, \phi_2 : F \to M$ which are δ-homomorphisms in $\|\cdot\|_2$ and $\|\tau \circ \phi_1 - \tau \circ \phi_2\|_2 < \delta$ are ϵ-approximately unitarily equivalent, i.e. there exists $v \in U(M)$ such that $\|v\phi_1(x)v^* - \phi_2(x)\|_2 < \epsilon$, when $\|x\| \leqslant 1$.*

Proposition 3.3. *Let M denote II_1 factor and $B \subseteq M$ be its von Neumann subalgebra. Then there exists the unique trace preserving conditional expectation $\Phi : M \to B$.*

sketch of a proof. Conditional expectation Φ can be defined by the following commutative diagram

$$\begin{array}{ccc} L^2(M) & \xrightarrow{\textit{projetion}} & L^2(B) \\ \text{\rotatebox{90}{$\subseteq$}} & & \text{\rotatebox{90}{$\subseteq$}} \\ M & \xrightarrow{\Phi} & B \end{array}$$

where M nad B are considered as their completions. Existence of Φ follows from arguments based on the double commutant theorem. □

Remark 3.4. Conditional expectation Φ is not a homomorphism.

Remark 3.5. In particular the last proposition show that injectivity is transferred to subalgebras (in fact it is not only true for II_1 factor, but also for any finite von Neumann algebra).

We now ready to present the proof of proposition 1.19, i.e. we will show the uniqueness of hyperfinite II_1 factor R. the following reasoning is an example of intertwining technique.

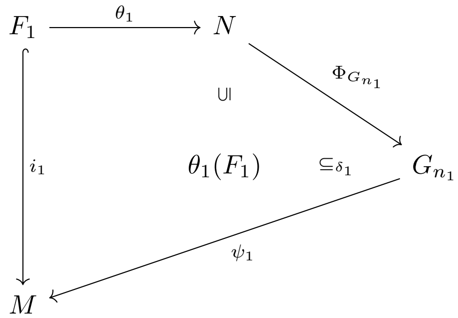
sketch of a proof. Suppose that we have two hyperfinite II_1 factors $M = \overline{\bigcup_n F_n}^{SOT}$ and $M = \overline{\bigcup_n G_n}^{SOT}$. Let us start with F_1 being subalgebra of M. There is a *-homomorphism

$$\theta_1 : F_1 \to N$$

induced by the trace in M. If so then there exists $G_{n_1} \subseteq N$ such that $\theta_1(F_1) \subseteq_{\delta_1} G_{n_1}$ ($\theta_1(F_1)$ is almost contained in G_{n_1} with respect to $\|\cdot\|_2$ norm, i.e. every element form the $\|\cdot\|$ norm unit ball of $\theta_1(F_1)$ is approximated by elements from G_{n_1} in $\|\cdot\|_2$ norm). As in a previous case, trace on N induces *-homomorphism

$$\psi_1 : G_{n1} \to M$$

and there is $F_{n_2} \subseteq M$ such that $\psi_1(G_{n_2}) \subseteq_{\delta_2} F_{n_2}$. Let us observe than when $\Phi_{G_{n_1}} : N \to G_{n_1}$denotes the conditional expectation, composition $\psi_1 \circ \Phi_{G_{n_1}} \circ \theta_1$ define an approximate *-homomorphism. Consider inclusion $i_1 : F_1 \to M$. Since i_1 and $\psi_1 \circ \Phi_{G_{n_1}} \circ \theta_1$ approximately agree on traces therefore, by uniqueness theorem, we can adjust ψ_1 by some unitaries in such a way that the following diagram



ϵ_1-approximately commutes (in fact value of δ_1 is dependent of a desired ϵ_1, for a given $\epsilon_1 > 0$ we find n_1 big enough in order to provide $\theta_1(F_1) \subseteq_{\delta_1} G_{n_1}$). By the same reasoning one can define $\theta_2 : F_{n_2} \to N$ induced by trace in M and continue this procedure (adjusting on maps on each step in order to obtain approximately commutative diagrams). Carrying on averaging for summable tolerances we obtain $\theta(x) = \lim \theta_n(x)$ (in $\|\cdot\|_2$ norm) as a map

$$\theta : \bigcup_n F_n \to N.$$

Taking the unique extension of θ to the whole M be obtain a *-homomorphism. Similarly, we define $\psi : N \to M$. Note that $\theta = \psi^{-1}$ so M and N are isomorphic. $\square$

Following the same strategy one can also prove the classification theorem for AF algebras (by ordered K_0). Let us remind the construction of K_0 group for a given C*-algebra A. By

$$\mathcal{P}_A = P\left(\bigcup_n M_n(A)\right)/_\sim$$

we denote the set of equivalence classes of projections in $\bigcup_n M_n(A)$, where $\sim$ denote Murray-von Neumann equivalence of projections. We can equipped $\mathcal{P}_A$ with abelian semigroup structure given by

$$[p] + [q] = \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}\right].$$

$K_0(A)$ is then define as a Grothendieck group constructed form $\mathcal{P}_A$. Consider a stably finite case: $M_n(A)$ is finite (where the notion of finiteness coincides with definition for von Neumann algebras, cf. section 1). $K_0(A)$ has a cone with order described by

$$K_0(A)_+ = \{[p] : p \in \mathcal{P}_A \subset K_0(A)\}.$$

After this introduction we are ready to presents the desired classification theorem.

Theorem 3.6 (Elliott)**.** *Two AF algebras A nad B are isomorphic if and only if they agree on the level of invariant*

$$(K_0(A), K_0(A)_+, [\mathbb{1}_A]) \cong (K_0(B), K_0(B)_+, [\mathbb{1}_B]),$$

i.e. there exists an isomorphism $\Theta : K_0(A) \to K_0(B)$ such that

$$\Theta(K_0(A)_+) = K_0(B)_+$$

and

$$\Theta([\mathbb{1}_A]) = [\mathbb{1}_B].$$

sketch of a proof. Suppose we have an isomorphism $\Theta : (K_0(A), K_0(A)_+, [\mathbb{1}_A]) \to (K_0(B), K_0(B)_+, [\mathbb{1}_B])$. Since K_0 is continuous, i.e. for $A = \overline{\bigcup_n A_n}$ we have that K_0 is a direct limit

$$(K_0(A), K_0(A)_+, [\mathbb{1}_A]) = \varinjlim (K_0(A_n), K_0(A_n)_+, [\mathbb{1}_{A_n}]),$$

we obtain the following diagram by restricting Θ to K-theory invariants of a given finite dimensional subalgebra of A or B

$$
\begin{array}{ccc}
(K_0(A_1), K_0(A_1)_+, [\mathbb{1}_{A_1}]) & \xrightarrow{\Theta} & \big(K_0(B_{n_1}), K_0(B_{n_1})_+, [\mathbb{1}_{B_{n_1}}]\big) \\
\big\downarrow & \overset{\Theta^{-1}}{\swarrow} & \big\downarrow \\
\big(K_0(A_{n_2}), K_0(A_{n_2})_+, [\mathbb{1}_{A_{n_2}}]\big) & \xrightarrow{\Theta} & \big(K_0(B_{n_3}), K_0(B_{n_3})_+, [\mathbb{1}_{B_{n_3}}]\big) \\
\big\downarrow & \overset{\Theta^{-1}}{\swarrow} & \big\downarrow \\
\dots & \xrightarrow{\Theta} & \dots
\end{array}
.
$$

The proof is then completed by existence and uniqueness theorem for finite dimensional algebras. □

Remark 3.7. If there is an isomorphism $\phi : A \to B$, then it induces the desired group isomorphism Θ on the level of K-theory. Conversely, if there is an isomorphism Θ then it can be lifted to isomorphism $\phi : A \to B$ such that its induced map on the level of K-theory agrees with Θ.

4 Connes's proof: injectivity implies hyperfiniteness

We will now return to the theorem 1.15. There are three main ingredients of Connes's proof that injectivity implies hyperfinitenss. Let M denote separable acting II_1 factor.

Ingredient 1 Uniqueness for commuting *-homomorphisms, i.e. M has approximately inner flip: There is a net u_n of unitary elements such that $u_n(x \otimes y)u_n^* \to y \otimes x$ in $\|\cdot\|_2$ norm for any $x, y \in M$.

Ingredient 2 M is McDuff (has McDuff property), i.e. $M \cong M\overline{\otimes}R$. In fact if M is McDuff then there exists $\Theta : M \cong M\overline{\otimes}R$ which is approximately unitarily equivalent to $\mathrm{id}_M \otimes \mathbb{1}_R$, i.e. there exists a net v_n of unitaries such that $v_n\Theta(x)v_n^* \to x \otimes x$ in $\|\cdot\|_2$ norm.

Ingredient 3 M has external finite dimensional approximation, i.e. $M \hookrightarrow R^\omega$, where R^ω denote ultraproduct of R defined by

$$
R^\omega = \ell^\infty(R) \Big/ \{(x_n) \in \ell^\infty(R) : \lim_{n \to \omega} \|x_n\|_2 = 0\}
$$

where $\omega \in \beta(\mathbb{N})/\mathbb{N}$ is notrivial ultrafilter.

Remark 4.1. Injectivity of M implies ingredients $1 - 3$.

We can now present the sketch of the proof that injectivity implies hyperfiniteness, assuming (for simplicity) that $M \hookrightarrow R$ (instead of $M \hookrightarrow R^\omega$).

sketch of a proof. Take finite subset $\mathcal{F} \ll M$ and $\epsilon > 0$. Let $F \subseteq M$ be a finite subalgebra such that $\mathcal{F} \subseteq_\epsilon F$ (with respect to $\|\cdot\|_2$ norm). Because M is McDuff, we can work with $M\overline{\otimes}R$ instead of M. If so then $\mathcal{F}$ is of the form $\{x_1 \otimes \mathbb{1}, \dots, x_n \otimes \mathbb{1}\}$. Consider the following maps

$$M \xrightarrow{\psi} M\overline{\otimes}R \xrightarrow{\mathrm{id}\otimes\phi} M\overline{\otimes}R$$

with ϕ being flip and ψ denoted either $x \mapsto x \otimes \mathbb{1}$ or $x \mapsto \mathbb{1} \otimes x$. Because of approximate unitary equivalence, there exist unitary $v \in M\overline{\otimes}R$ such that $\|v(\mathbb{1} \otimes \phi(x_i))v^* - x_1 \otimes \mathbb{1}\|_2 < \epsilon$. There is a finite set $\{\phi(x_1), \dots, \phi(x_n)\}$ and a finite dimensional subalgebra $\tilde{F} \subseteq R$ such that $\{\phi(x_1), \dots, \phi(x_n)\} \subseteq_\epsilon \tilde{F}$. Finally, we obtain finite dimensional subalgebra $v(\mathbb{1}\overline{\otimes}\tilde{F})v^* \subseteq M\overline{\otimes}R$ with $x_i \otimes \mathbb{1} \in_{k\epsilon} v(\mathbb{1}\overline{\otimes}\tilde{F})v^*$ for some $k \in \mathbb{N}$. □

It is worth noting that there is a direct C*-algebraic analogue of that reasoning.

Theorem 4.2 (Effros-Rosenberg, 1978)**.** *If A is separable unital C*-algebra such that*

1. *A has approximately inner flip (in this case with respect to $\|\cdot\|$ norm),*
2. *$A \cong A \otimes Q$ (Q is a universal UHF algebra),*
3. *$A \hookrightarrow Q_\omega$ (Q_ω denotes C*-ultraproduct).*

Then A is AF algebra (and hence $A \cong Q$ by classification).

Remark 4.3. Not all separable, unital C*-algebras fulfill conditions $1-3$ stated in theorem 4.2 (for example CAR algebra).

Theorem 4.4. *If C*-algebra A has approximately inner flip then A is simple, nuclear and has unique trace of rank* 1 *(if it exists).*

sketch of a proof. Suppose that $J \lhd A$ is a proper ideal (closed, two-sided). Then $J \otimes A$ and $A \otimes J$ are different ideals in $A \otimes A$. Since A has approximately inner flip there are unitaries u_n such that $u_n(J \otimes A)u_n^* \to J \otimes A$ and hence $J \otimes A = A \otimes J$, because unitaries preserve ideals. We get a contradiction thus A is simple.

To see that A is nuclear consider the flip $A \otimes_{\max} (A \otimes B) \xrightarrow{\text{flip}} A \otimes A \otimes B$. We have unitaries u_n such that $u_n(\mathbb{1} \otimes_{\max} (A \otimes B))u_n^* \to A \otimes \mathbb{1} \otimes B$ which shows that maximal norm and minimal norm coincides.

The remaining part connected to trace is left as an exercise. □

Remark 4.5. Having approximately inner flip implies that flip is trivial on the level of K-theory, e.g. AF algebra ⇒ UHF algebra. Therefore, condition 1 in theorem 4.2 excludes broad range of C*-algebras.

Theorem 4.6 (Matui, Sato)**.** *Suppose A is simple, separable, nuclear unital C*-algebra with the unique trace and such that $A \cong A \otimes Q$. Then A has a "2-colored approximately inner flip"', i.e there exists net $\{(u_n, v_n)\}_n$ of pairs of unitaries contractions such that*

$$v_n(x \otimes y)v_n^* + u_n(x \otimes y)u_n^* \to y \otimes x$$

and $u_n^*u_n$, $v_n^*v_n$ approximately commutes, i.e. $[u_n^*u_n, A\otimes A]\to 0$ and $[v_n^*v_n, A\otimes A]\to 0$.

Corollary 4.7 (Matui, Sato). *Let A be simple, separable, unital C*-algebra with the unique trace. Suppose that $A\cong A\otimes Q$ and $A\hookrightarrow Q_\omega$. Then $\dim_{\mathrm{nuc}}(A)\leqslant 1$.*

5 Quasidiagonality

In this section we will briefly discuss quasidiagonality which has already appeared in our previous considerations. We shall begin with the following definition.

Definition 5.1. We say that nuclear, separable, unital C*-algebra is quasidiagonal if $A\hookrightarrow Q_\omega$. This is the case if and only if there exists a sequence of maps $\phi_n : A\to M_{k_n}(\mathbb{C})$ such that

$$\|\phi_n(xy)-\phi_n(x)\phi_n(y)\|\to 0$$

and $\|\phi_n(x)\|\to\|x\|$ for any $x,y\in A$.

Example 5.2. The following C*-algebras are quasidiagonal:

- Abelian C*-algebras,
- AF algebras,
- $C(X,M_n)$.

Obstruction: $A\hookrightarrow Q_\omega$, where

$$Q_\omega=\ell^\infty(Q)\Big/\{(x_n)\in\ell^\infty(A):\lim_{n\to\omega}\|x_n\|=0\},\ \omega\in\beta(\mathbb{N})/\mathbb{N}$$

implies that A is stably finite.

Open problem (Blackadar, Kirschberg): Do all stably finite nuclear C*-algebras A fulfill $A\hookrightarrow Q_\omega$?

Theorem 5.3 (Winter, Zacharias). *If A is separable, nuclear an unital C*-algebra with faithful trace, then there exist contractive order zero maps $\phi_1,\phi_2 : A\to Q_\omega$ such that $\phi_1(\mathbb{1})+\phi_2(\mathbb{1})=1$.*

Corollary 5.4 (Sato, White, Winter). *Let A be a simple, separable, nuclear, unital C*-algebra with unique trace such that $A\cong A\otimes Q$ (or $A\cong A\otimes Z$). Then $\dim_{\mathrm{nuc}}(A)\leqslant 3$.*

Theorem 5.5 (Tikuisis, White, Winter). *A separable, unital, nuclear C*-algebra A with faithful trace and UCT admits embedding $A\hookrightarrow Q_\omega$.*

Remark 5.6. The only role of the presence of UCT in the previous theorem is to provide that the map

$$KK\left(C_0(0,1)\otimes A,\prod_{n=1}^{\infty}Q_\omega\right)\to\prod_{n=1}^{\infty}KK\left(C_0(0,1)\otimes A,Q_\omega\right)$$

is injective.

6 Tracial approximation

Let us start with the natural question. Whenever there are two linear maps $\phi, \psi : A \to B$ between unital C*-algebras, when are they unitarily equivalent or at least approximately unitarily equivalent? The answer for that question is provided by Connes 2×2 matrix trick. Define a map

$$\pi : A \to \begin{pmatrix} \phi(x) & 0 \\ 0 & \psi(x) \end{pmatrix}.$$

We have the following facts

Fact 6.1. *ϕ is unitarily equivalent to ψ if and only if matrices*

$$\begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}$$

are equivalent (in Murray-von Neumann sense) in $M_2(B) \cap \pi(A)'$

Fact 6.2. *ϕ is approximately unitarily equivalent to ψ if and only if matrices*

$$\begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}$$

are equivalent (in Murray-von Neumann sense) in the relative commutant of $\pi(A)$ in the ultraproduct of $M_2(B)$.

It is immediate that any two maps form R (hyperfinite II_1 factor) into other II_1 factor are approximately unitarily equivalent (because unitary equivalence occur on finite dimensional algebras). One could say even more. If $\pi : R \to M^\omega$ then $\pi(R)' \cap M^\omega$ is a factor of type II (trace on M^ω is inherited from trace on M).

Example 6.3. Consider the AF algebra A with $K_0(A) = \mathbb{Q}^2$, $K_0(A)_+ = \{(q,p) \in \mathbb{Q}_+ \times \mathbb{Q}\}$ and $[\mathbb{1}] = (0, 1)$. Traces of AF algebras comes from states in K-theory, i.e. from maps

$$(K_0(A_1), K_0(A_1)_+, [\mathbb{1}_{A_1}]) \to (\mathbb{R}, \mathbb{R}_+, 1).$$

In a given example there is only one state ρ mapping

$$(s,t) \mapsto s.$$

This state has the property that $[x] \leqslant [y]$ implies $\rho(x) \leqslant \rho(y)$ (but the inverse implication does not occur).

Definition 6.4. C*-algebra A has strict comparison of projections by traces if $\tau(p) < \tau(q)$ for all $\tau \in T(A)$ $(p, q \in P(\bigcup_n M_n(A)))$ implies $[p] \leqslant [q]$.

Proposition 6.5. *Let A and B be separable C*-algebras with the unique trace. Let $\pi : A \to B$ be* *a *-homomorphism. By $\overline{\pi}$ we denote extension of π*

$$\overline{\pi} : M \to N$$

where $M = \overline{A}^{\mathrm{SOT}}$ and $N = \overline{B}^{\mathrm{SOT}}$. Then we have a surjective map (form Kaplansky density theorem) between ultraproducts

$$B_\omega \twoheadrightarrow N^\omega$$

which induces a surjective map

$$\pi(A)' \cap B_\omega \twoheadrightarrow \overline{\pi}(M)' \cap N^\omega.$$

This proposition gives us a tool of "pulling back" elements form von Neumann setting ($\overline{\pi}(M)' \cap N^\omega$) into the C*-algebraic one ($\pi(A)' \cap B_\omega$).

In the remaining part of this section we shall assume that each C*-algebra is separable, simple, nuclear and with the unique trace.

Lemma 6.6 (Matui, Sato)**.** *Suppose that B has strict comparison of elements by its trace. Then*

1. *all traces on $\pi(A)' \cap B_\omega$ factor through canonical surjection onto $\overline{\pi}(M)' \cap N^\omega$, i.e. $\pi(A)' \cap B_\omega$ has unique trace.*

2. *$\pi(A)' \cap B_\omega$ also has strict comparison.*

We have already defined strict comparison in the context of projections, the following definition provides its generalization.

Definition 6.7. A has strict comparison if for any positive elements a, b in matrix algebra $\bigcup_n M_n(A)$, $\lim_n \tau(a^{\frac{1}{n}}) < \lim_n \tau(b^{\frac{1}{n}})$ implies that there exists a sequence of unitaries v_n such that $v_n b v_n^* \to a$.

Assume A is unital. What is more let $A \hookrightarrow Q_\omega$ (holds in the presence of UCT) and $A \cong A \otimes Q$. Fix unital completely positive maps $\varphi_n : A \to M_{k_n} \subset M_{k_n} \otimes Q \cong Q$ inducing $\varphi : A \otimes Q \hookrightarrow Q_\omega$. Fix projections $p_n = \mathbb{1} \otimes \overline{p_n} \in M_{k_n} \otimes Q$ such that $1 - \epsilon \leqslant \tau(p_n) \leqslant 1 - \frac{\epsilon}{2}$ and define projection $p = (p_n) \in Q_\omega \cap \varphi(A \otimes Q)'$ (we have $1 - \epsilon \leqslant \tau(p) \leqslant 1 - \frac{\epsilon}{2}$). Set $\pi : A \otimes Q \to M_2(A \otimes Q)_\omega$ by

$$\pi(x) = \begin{pmatrix} x & 0 \\ 0 & \mathbb{1}_A \otimes \varphi(x). \end{pmatrix}$$

By lemma 6.6, $\pi(A \otimes Q)' \cap M_2(A \otimes Q)_\omega$ has strict comparison

$$\tau\left(\begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix}\right) < \tau\left(\begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}\right) \Rightarrow \exists v \in \pi(A \otimes Q)' \cap M_2(A \otimes Q)_\omega \text{ s.t.}$$

$$v^* v = \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix}, \; vv^* \leqslant \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $F_n = M_{k_n} \otimes \overline{p_n}$ lift v to (v_n) (in the ultraproduct) satisfying

$$v_n^* v_n = \begin{pmatrix} 0 & 0 \\ 0 & p_n \end{pmatrix}, \; vv^* = \begin{pmatrix} q_n & 0 \\ 0 & 0 \end{pmatrix} \leqslant \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Note that $[q_n, A \otimes Q] \to 0$.

Conclusion: $A \otimes Q$ is tracially AF in this setting, i.e. for any finite subset $\mathcal{F} \ll A$, $\epsilon > 0$ there exists finite dimensional subalgebra $F \subset A \otimes Q$ such that $\|[\mathbb{1}_F, x]\| < \epsilon$ for any $x \in \mathcal{F}$, $\tau(\mathbb{1}_F) > 1 - \epsilon$ and $\mathbb{1}_F x \mathbb{1}_F \in_\epsilon F$ for all $x \in \mathcal{F}$.

This tracial local structure is good for classification in the presence of UCT. Unfortunately tensoring with Q "damage" K-theory information. The proof of presented construction give us opportunity to change Q with any UHF algebra.

7 Z-stability

In this section we will return to the main idea presented by the table in the end of section 2. Let us recall that in a von Neumann setting we have

$$\text{McDuff property } M \overline{\otimes} R \cong M \Leftrightarrow \forall_k M_k \hookrightarrow M^\omega \cap M' (\text{unitarily}).$$

Similarly in C*-algebraic setting

$$\text{McDuff property } A \otimes Q \cong A \Leftrightarrow \forall_k M_k \hookrightarrow A_\omega \cap A' (\text{unitarily}).$$

Let Z denotes (as previous) the Jiang-Su C*-algebra. We have the following dimensional ("colored") version of above results.

Proposition 7.1. *$A \cong A \otimes Z$ if and only if for any k there exist contractive order zero maps*

$$\phi_1 : M_k \to A_\omega \cap A',$$

$$\phi_2 : M_{k+1} \to A_\omega \cap A'$$

such that $\phi_1(\mathbb{1}) + \phi_2(\mathbb{1}) = \mathbb{1}$.

Remark 7.2. In fact this property could be seen as a definition of Jiang-Su C*-algebra.

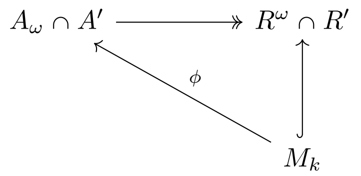
Remark 7.3. Most of proofs from the previous sections have their colored versions (with $A \otimes Z \cong A$ instead of $A \otimes Q \cong A$).

The above proposition can be stated in a different way.

Proposition 7.4. *$A \cong A \otimes Z$ if and only if for any k there exists an order zero map $\phi : M_k \to A_\omega \cap A'$ such that $\mathbb{1} - \phi(\mathbb{1}) = v^* v$ and $\phi(e_{11}) vv^* = vv^*$ for some $v \in A_\omega \cap A'$.*

Theorem 7.5 (Matui, Sato)**.** *If A is a unital, simple, separable, nuclear C*-algebra with the unique trace then $A \cong A \otimes Z$.*

sketch of a proof. We have the following diagram from canonical surjection



where ϕ denotes order zero lift and inclusion of M_k in $R^\omega \cap R'$ is by McDuff property. Observe that $\tau(\mathbb{1} - \phi(\mathbb{1})) = 0$ and $\tau(\phi(e_{11})) = \frac{1}{k}$, so there exists desired v from the previous definition.

□

Remark 7.6. Let us emphasis that Z was a first counterexample to the conjecture that simple nuclear and separable C*-algebras are completely classified by K-theory. Indeed, Z is stably finite and infinite dimensional C*-algebra with K-theory $K_*(Z) = K_*(\mathbb{C})$.

Finally, let us present more direct approach to the definition of Jiang-Su algebra. Let

$$Z_{p^\infty,q^\infty} = \{f \in C([0,1], M_{p^\infty} \otimes M_{q^\infty}) : f(0) = M_{p^\infty} \otimes \mathbb{1},\ f(1) = \mathbb{1} \otimes M_{q^\infty}\}.$$

Remark 7.7. One can think of this as a C*-algebraic analogue of join construction.

Consider a map

$$\alpha : Z_{p^\infty,q^\infty} \to Z_{p^\infty,q^\infty}$$

which is trace collapsing, i.e. $\tau_1 \circ \alpha = \tau_2 \circ \alpha$ for all traces $\tau_1, \tau_2 \in Z_{p^\infty,q^\infty}$. Then we can define Z (knowing of its existence form elsewhere) by

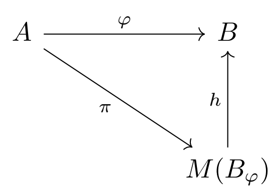
$$Z_{p^\infty,q^\infty} \xrightarrow{\alpha} Z_{p^\infty,q^\infty} \xrightarrow{\alpha} Z_{p^\infty,q^\infty} \xrightarrow{\alpha} \ldots \xrightarrow{\alpha} Z.$$

"Winter technique": If one can classify $A \otimes U$ for all $U \in$ UHF algebras in a fashion compatible with the definition of Z, then one can classify $A \otimes Z$.

8 Nuclear dimension: examples, properties, techniques

In this section we discuss in more detailed way the notion of nuclear dimension which was briefly presented in section 2. Recall than a completely positive map $\varphi : A \to B$ between two C*-algebras is order zero (oz) if $a_1 a_2 = 0$ ($a_1 \perp a_2$) implies $\varphi(a_1)\varphi(a_2) = 0$ ($\varphi(a_1) \perp \varphi(a_2)$) for all $a_1, a_2 \in A_+$.

Theorem 8.1. *A map $\varphi : A \to B$ is order zero if and only if we have the factorization given by the following commutative diagram*



*where π is a *-homomorphism form A to multiplier algebra $M(B_\varphi)$, h is some positive element of B such that $[h, \pi(A)] = 0$ and $B_\varphi = \overline{\varphi(A)B\varphi(A)}$. We have $\varphi(a) = h\pi(a) = \pi(a)h = h^{\frac{1}{2}}\pi(a)h^{\frac{1}{2}}$.*

Remark 8.2. If A is unital then $h = \varphi(\mathbb{1})$. In general case one should deal with approximate unit.

Before we go further let us evoke two important stability results.

Lemma 8.3. *Let A be a C*-algebra with closed two-sided ideal $I \lhd A$ and F be a finite dimensional C*-algebra. If there exists an order zero map $\varphi : F \to A/I$, then there exists an order zero lift $\tilde{\varphi} : F \to A$.*

Let $\varphi : F \to A$ be a completely positive map form finite dimensional C*-algebra F to some C*-algebra A. We define a modulus of order zero map by

$$\delta(\varphi) = \sup \{\|\varphi(x)\varphi(y)\| : x, y \in F_+,\ \|x\|, \|y\| \leqslant 1,\ x \perp y\}.$$

It is obvious that $\delta(\varphi) = 0$ if and only if φ is order zero.

Lemma 8.4. *(Perturbation result) For any $\epsilon > 0$ there exists $\delta > 0$ such that for all $\varphi : F \to A$ completely positive contractions with $\delta(\varphi) < \delta$ we can find an order zero map $\tilde{\varphi} : F \to A$ such that $\|\varphi - \tilde{\varphi}\| < \epsilon$.*

Recall (definition 2.14) that C*-algebra has nuclear dimension $\dim_{\text{nuc}}(A) \leqslant n$ if and only if for any finite subset $\mathcal{F} \ll A$ and any $\epsilon > 0$ there is a completely positive approximation

$$A \xrightarrow{\quad\psi\quad} F = F^{(0)} \oplus \ldots \oplus F^{(n)} \xrightarrow{\quad\varphi\quad} A$$

such that $\|\psi\| \leqslant 1$, $\varphi|_{F^{(i)}}$ is order zero contraction for all $i = 0, 1, \ldots, n$ and $\|\phi\psi(x) - x\| < \epsilon$ for any $x \in \mathcal{F}$. The above diagram give us so called n-decomposable approximation.

Basic properties:

1) $\dim_{\text{nuc}}(A) = 0$ if and only if A is AF C*-algebra.

2) $\dim_{\text{nuc}}(\lim_{i\to\infty} A_i) \leqslant \liminf_{i\to\infty} \dim_{\text{nuc}}(A_i)$.

3) $\dim_{\text{nuc}}(M_n(A)) = \dim_{\text{nuc}}(A) = \dim_{\text{nuc}}(A \otimes K)$ (Morita invariance).

sketch of a proof. Just tensor decomposable approximation diagram with M_n and then use property 2 stated above to prove the last equality. □

4) Consider the following exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0.$$

Then $\dim_{\text{nuc}}(I), \dim_{\text{nuc}}(A/I) \leqslant \dim_{\text{nuc}}(A) \leqslant \dim_{\text{nuc}}(I) + \dim_{\text{nuc}}(A/I) + 1$.

sketch of a proof. To show $\dim_{\mathrm{nuc}}(I) \leqslant \dim_{\mathrm{nuc}}(A)$ consider the following diagram

$$\begin{array}{ccccccc} I & \hookrightarrow & A & \xrightarrow{\mathrm{id}_A} & A & \xrightarrow{\phi} & I \\ & & & \searrow^{\psi} & \uparrow{\scriptstyle \varphi} & & \\ & & & & F = F^{(0)} \oplus \ldots \oplus F^{(n)} & & \end{array}$$

with $\phi : a \mapsto e_\lambda^{\frac{1}{2}} a e_\lambda^{\frac{1}{2}}$ given by quasicentral approximate unit $(e_\lambda) \in I$ (we say that approximate unit of an ideal $I \lhd A$ is quasicentral if $[e_\lambda, a] \to 0$ for all $a \in A$). Observe that $\phi\varphi|_{F^{(i)}} : F^{(i)} \to I$ is approximately order zero so using lemma 8.4 one can perturbate it into order zero map.

Inequality $\dim_{\mathrm{nuc}}(A/I) \leqslant \dim_{\mathrm{nuc}}(A)$ comes from the following diagram

$$\begin{array}{ccccccc} A/I & \xrightarrow{ce} & A & \xrightarrow{\mathrm{id}_A} & A & \xrightarrow{q} & A/I \\ & & & \searrow^{\psi} & \uparrow{\scriptstyle \varphi} & & \\ & & & & F = F^{(0)} \oplus \ldots \oplus F^{(n)} & & \end{array}$$

where q denotes quotient map and ce stands for Choi-Effros cp map.

Finally, in order to show $\dim_{\mathrm{nuc}}(A) \leqslant \dim_{\mathrm{nuc}}(I) + \dim_{\mathrm{nuc}}(A/I) + 1$ consider the diagram which consists of decomposable approximation diagrams for I and A/I

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & A/I & \longrightarrow & 0 \\ & & \downarrow & & & & \downarrow & & \\ & & F^{(0)} \oplus \ldots \oplus F^{(n)} & & \bigoplus & & G^{(0)} \oplus \ldots \oplus G^{(m)} & & \\ & & \downarrow & & & & \downarrow & & \\ 0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & A/I & \longrightarrow & 0 \end{array}$$

Using once more quasicentral approximated unit trick and taking the direct sum of $F^{(0)} \oplus \ldots \oplus F^{(n)}$ and $G^{(0)} \oplus \ldots \oplus G^{(m)}$ we get the thesis. □

5) Let $H \subseteq A$ be a hereditary subalgebra of a C*-algebra A (i.e. for any $x \in H$ and $y \in A$ $y \leqslant x$ implies $y \in H$ - if A is separable then hereditary subalgebra is of the form $H = \overline{aAa}$ for some positive element a). Then $\dim_{\mathrm{nuc}}(H) \leqslant \dim_{\mathrm{nuc}}(A)$.

6) $\dim_{\mathrm{nuc}}(A \otimes B) \leqslant (\dim_{\mathrm{nuc}}(A) + 1)(\dim_{\mathrm{nuc}}(B) + 1) - 1$. Putting $\dim^+_{\mathrm{nuc}}(A) = \dim_{\mathrm{nuc}}(A) + 1$ we can write it in a simpler way $\dim^+_{\mathrm{nuc}}(A \otimes B) \leqslant (\dim^+_{\mathrm{nuc}}(A))(\dim^+_{\mathrm{nuc}}(B))$.

sketch of a proof. Just take tensor product of decomposable approximation diagrams. □

7) Technical point: If $\dim_{\mathrm{nuc}}(A) \leqslant n$ and $\mathcal{F} \Subset A$ is a finite subset then there exists n-decomposable approximation such that ψ is approximately order zero map on $\mathcal{F}$. If $\|\varphi\| = 1$ we can arrange for ψ to be approximately multiplicative (in that situation A is quasidiagonal).

We are now ready to provide some concrete examples of nuclear dimensions.

Definition 8.5. Let X be a compact Hausdorff space. X has decomposition dimension $\dim(X) \leqslant n$ if for any (finite) open cover W there exists an open refinement $U > W$ (i.e. for any $U^i \in U$ there exists $W^i \in W$ such that $U^i \subset W^i$ and U is a cover) such that $U = U^{(0)} \cup U^{(1)} \cup \ldots \cup U^{(n)}$, where each $U^{(i)}$ is a union of pairwise disjoint open sets.

Remark 8.6. Decomposition dimension is the same as usual cover dimension (in most reasonable cases).

Example 8.7. If X is compact (metrizable) Hausdorff space then $\dim_{\text{nuc}}(C(X)) = \dim(X)$.

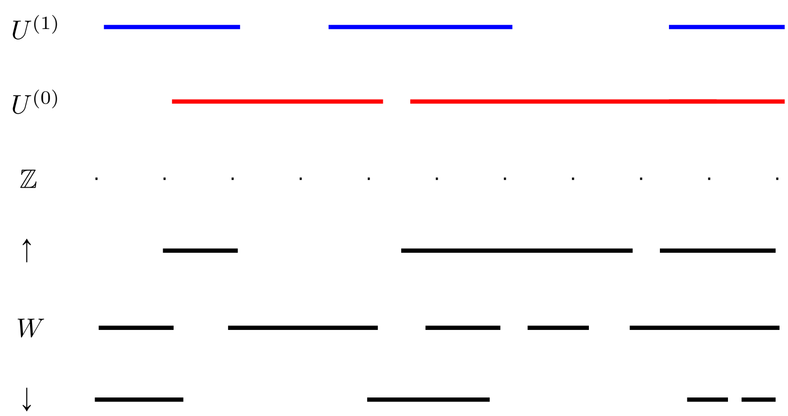
sketch of a proof. To show that $\dim_{\text{nuc}}(C(X)) \leqslant \dim(X)$ one should generalized discussion given in example 2.17, i.e. partition of unit subordinated to $U = U^{(0)} \cup U^{(1)} \cup \ldots \cup U^{(n)}$ defines n-decomposable approximation. In order to prove the second inequality one should start with n-decomposable approximation for $C(X)$ and observe that existence of order zero maps from matrix algebra to commutative C*-algebras implies that considered matrix algebras are one dimensional. □

Example 8.8 (Roe algebra)**.** Let (X, d) be a discrete metric space of bounded geometry, i.e. for any $r > 0$ number of elements $|B_r(x)|$ in ball of radius r and center at x is uniformly bounded ($d_r = \sup\{|B_r(x)| : x \in X\} < \infty$). In this case we define finite propagation operators by

$$UC(X) = \{[\alpha_{x,y}]_{x,y\in X} : \exists M > 0, R > 0 \text{ s.t. } |\alpha_{x,y}| \leqslant M \text{ and } \alpha_{x,y} = 0 \text{ if } d(x,y) > R\}.$$

For example if $X = \mathbb{Z}$ then we consider infinite matrices with entries different than zero only in the R-width bar across the diagonal. The Roe algebra is defined by $UC^*_r(X) = \overline{UC(X)} \subseteq B(\ell^2(X))$.

Because X is discrete, its topological dimension is equal to zero. However, one can define new dimensional notion suitable for a coarse space. Space X has asymptotic dimension $\operatorname{asdim}(X) \leqslant n$ if and only if for any uniform cover W (diameter $d(w)$ of $w \in W$ is uniformly bounded) there exists a coarsening U ($W > U$) such that $U = U^{(0)} \cup U^{(1)} \cup \ldots \cup U^{(n)}$, where each $U^{(i)}$ is a union of pairwise disjoint sets (even r-disjoint for some $r > 0$).



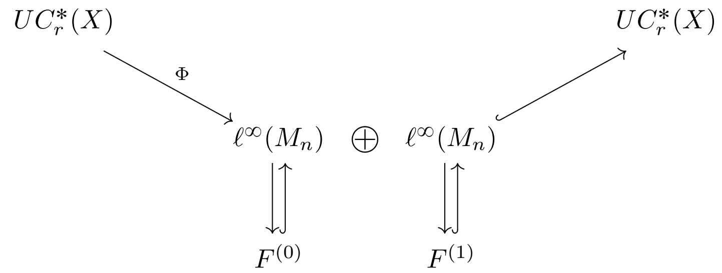
From the above figure it is clear that $\operatorname{asdim}(\mathbb{Z}) = 1$. By similar reasoning one can obtain also $\operatorname{asdim}(\mathbb{Z}^d) = d$.

In the case of Roe algebra $UC_r^*(X)$ we have the following inequality.

Theorem 8.9. *Nuclear dimension of Roe algebra satisfies* $\dim_{\text{nuc}}(UC_r^*(X)) \leqslant \operatorname{asdim}(X)$.

Remark 8.10. It is interesting results since Roe algebra is often quite big (i.e. nonseparable).

sketch of a proof. We give argumentation in the case when $X = \mathbb{Z}$. Since $\ell^\infty(M_n)$ is AF algebra we have the following diagram



where $\Phi : UC_r^*(X) \to \ell^\infty(M_n) \bigoplus \ell^\infty(M_n)$ denotes cut down map which sends any element form $UC_r^*(X)$ to sequence of its submatrices (related to given family $U^{(0)}$ and $U^{(1)}$) placed in the first and second components of $\ell^\infty(M_n) \bigoplus \ell^\infty(M_n)$ respectively. Form this diagram we can construct desired approximation, the only problem is the fact that submatrices related to different families $U^{(0)}$ and $U^{(1)}$ may overlap. To get rid of this, each block matrix in any considered element form $\ell^\infty(M_n)$ must be rescaled (multiplied form left and right on the level of definition of the cut down map Φ) by the matrix of the form

$$D = \begin{pmatrix} \frac{2}{n} & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & \frac{2}{n} \end{pmatrix}^{\frac{1}{2}}.$$

Since D almost commutes with finite propagation operators the proof is completed. □

Definition 8.11. C*-algebra A is purely infinite and simple if for all $a, b \in A_+ \setminus \{0\}$ there exists $x \in A$ such that $a = x^* b x$ (if A is unital then $\mathbb{1} = x^* b x$).

Example 8.12. By arguments similar to the given in a previous example one can show that for Cuntz algebras we have $\dim_{\text{nuc}}(O_n) = 1$ and $\dim_{\text{nuc}}(O_\infty) \leqslant 2$. It quite unexpected result as Cuntz algebras are purely infinite. In fact all infinite graph C*-algebra have finite nuclear dimension.

Theorem 8.13 (Winter, Zacharias)**.** *If A is purely infinite, simple, separable and nuclear C*-algebra (Kirchberg algebra) satisfying UCT (Universal Coefficients Theorem), then* $\dim_{\text{nuc}}(A) \leqslant 5$.

In fact above resultrs has been improved.

Remark 8.14. $\dim_{\text{nuc}}(A) = 1$ for all Kirchberg algebras with the presence of UCT (Sims).

Remark 8.15. $\dim_{\text{nuc}}(A) \leqslant 3$ for all Kirchberg algebras without the presence of UCT (Matui, Sato).

Remark 8.16. $\dim_{\text{nuc}}(A) = 1$ for all Kirchberg algebras without the presence of UCT (Bosa, Brown, Sato, Tikuisis, White, Winter).

9 Cuntz semigroups and nuclear dimension

Let us consider C*-algebra A and define the direct limit $M_\infty(A) = \bigcup_n M_n(A)$ where the embedding of $M_n(A)$ in $M_{n+1}(A)$ is given by

$$M_n(A) \hookrightarrow \begin{pmatrix} M_n(A) & 0 \\ 0 & 0 \end{pmatrix}.$$

For $a, b \in M_\infty(A)_+$ we define relation $a \preceq b$ (we say that a is Cuntz below b) if there exists a sequence $(x_n) \subseteq M_\infty(A)$ such that $x_n^* b x_n \to a$ in norm. We say that a and b are Cuntz equivalent $a \sim b$ if and only if $a \preceq b$ and $b \preceq a$. Cuntz semigroup $W(A)$ is then define by $W(A) = M_\infty(A)_+/\sim$. Cuntz semigroup is ordered semigroup with addition given by

$$[a] + [b] = \left[\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right].$$

The importance of Cuntz semigroup $W(A)$ lies in the fact that it contains information of K-theory and traces of the starting algebra A. For example $K_0^*(A)$ - the Grothendieck group obtain form $W(A)$ corresponds to traces on A (in a unital case) with the exception of the case when A is purely infinite (in that situation all positive elements are Cuntz equivalent and $W(A) = \{0, \infty\}$). In general it is extremely hard to determine $W(A)$ (no good homological methods suitable for this problem).

One can think of $[a] \in W(A)$ in correspondence with open support projections (by open support we mean interior of support). For $a \in M_\infty(A)_+$ we have $a \sim a^n \sim a^{\frac{1}{n}}$. In the case of positive function $a \in C_0(\mathbb{R})$, $a^{\frac{1}{n}}$ converges (pointwise) to some open support projection $\chi_{\text{supp}(a)}$. In general case one can also make the above statement meaningful (this statement can be make precise on the level of enveloping von Neuman algebra of $A \otimes K$).

Let us observe that from trace on A we get trace on $M_\infty(A)$ $(W(A))$

$$d_\tau(a) = \tau([a]) = \lim_{n\to\infty} \tau\left(a^{\frac{1}{n}}\right).$$

Let us remind the definition 6.7 formulated in the present context.

Definition 9.1. C*-algebra A has strict comparison if for all $a, b \in M_\infty(A)_+ \setminus \{0\}$ the following is true: if $\tau(a) < \tau(b)$ for all traces $\tau \in T(A)$ then $a \preceq b$.

Example 9.2. The following examples give a glimpse of the previous definition.

1. Strict comparison is fulfilled in von Neumann algebras setting.
2. Let $A = M_n$ with the unique trace $\tau = \frac{1}{n}\mathrm{Tr}$ and p_a denotes the projection onto the range of $a \in A_+$. For any $a, b \in A_+$, $\tau(a) < \tau(b)$ implies rk $a <$ rk b and from that there exists $x \in A$ such that $a = x^*bx$, so we have strict comparison.
3. If simple and infinite (without trace) C*-algebra A has strict comparison then A is purely infinite ($a, b \in M_\infty(A)_+ \Rightarrow a \preceq b, b \preceq a, a \sim b$).
4. There exists an infinite, simple and nuclear C*-algebra without strict comparison (Rørdam, 2001).
5. There exists a finite, simple and nuclear C*-algebra without strict comparison (Toms, 2008).

To sum up, we have two notion of comparison (for positive elements):

- $a <_\tau b \Leftrightarrow \tau(p_a) < \tau(p_b)$ for all $\tau \in T(A)$,
- $a \preceq b \Leftrightarrow \exists (x_n) \subseteq A : x_n^* b x_n \to a$.

We have strict comparison if $[a] <_\tau [b]$ implies $[a] \preceq [b]$ ($a <_\tau b$ implies $a \preceq b$).

Theorem 9.3. *(Winter, Rørdam) A simple and separable C*-algebra A with* $\dim_{\mathrm{nuc}}(A) < \infty$ *has strict comparison.*

The proof of the previous theorem is very complicated ($\dim_{\mathrm{nuc}}(A) < \infty$ implies Z-stability which leads to strict comparison), but one can relatively easy prove the weaker version of this results given in the next proposition.

Proposition 9.4. *Suppose* $\dim_{\mathrm{nuc}}(A) = n < \infty$. *Then A has n-comparison, i.e. for any $a, b_0, b_1, \ldots, b_n \in M_\infty(A)_+$, condition $[a] <_\tau [b_0]$, $[a] <_\tau [b_1], \ldots,$ $[a] <_\tau [b_n]$ implies $[a] \preceq [b_0] + [b_1] + \ldots + [b_n]$. In particular $[a] <_\tau [b]$ implies $[a] \preceq (n+1)[b]$.*

Firstly, note that if $\varphi : A \to B$ is order zero map then $a_1 \preceq a_2$ implies $\varphi(a_1) \preceq \varphi(a_2)$. If $\tau \in T(B)$ then $\tau \circ \varphi \in T(A)$.

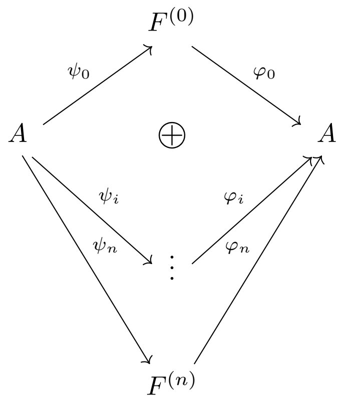
sketch of a proof. There is a sequence (x_n) such that $x_n a_2 x_n^* \to a_1$. If so, then $\varphi(x_n a_2 x_n^*) \to \varphi(a_1)$. By theorem 8.1 we have

$$\varphi(x_n^* a_2 x_n) = h\pi(x_n^* a_2 x_n) = h\pi(x_n^*)\pi(a_2)\pi(x_n) = \left(\pi(x_n)h^{\frac{1}{n}}\right)^* h^{1-\frac{2}{n}}\pi(a_2)\left(\pi(x_n)h^{\frac{1}{n}}\right)$$

and we are done because $h^{1-\frac{2}{n}}\pi(a_2) \to \varphi(a_2)$. Similarly $\tau \circ \varphi \in T(A)$ (by factorization of order zero map, cf. theorem 8.1). □

We are now ready to present the proof of proposition 9.4

sketch of a proof. Let $a <_\tau b_0, b_1, \ldots, b_n \in A_+(M_\infty(A)_+)$. Find n-decomposable approximation given by the following diagram



We can chose ψ_i for $i = 0, 1, \ldots, n$ to be sufficiently (approximately) order zero maps (φ_i are order zero maps from definition on n-decomposable approximation). We can then show that $\psi_i(a) <_\tau \psi_i(b_j)$ for $j = 0, 1, \ldots, n$ in $F^{(i)}$. Since we have a strict comparison in the case of finite dimensional C*-algebras it gives us $\psi_i(a) \precsim \psi_i(b_j)$. This implies $\varphi_i\psi_i(a) \precsim \varphi_i\psi_i(b_j) \precsim b_j$. Finally, we obtain $a \approx \sum_i \varphi_i\psi_i(a) \precsim \oplus\varphi_i\psi_i(a) \precsim \oplus\varphi_i\psi_i(b_j) \precsim \oplus b_j$ which ends the proof. □

Toms Winter Conjecture (2008): Let A be a separable, simple and nuclear C*-algebra. The following conditions are equivalent

1. $\dim_{\text{nuc}}(A) < \infty$.
2. A is Z-stable ($A \otimes Z \cong A$).
3. A has strict comparison.

Condition 1 could be seen as a kind of topological condition, while condition 2 is related to the analysis of the property described by Kirchberg (A nuclear, simple and $A \otimes O_\infty \cong A \Leftrightarrow A$ if and only if A is Kirchberg algebra - is purely infinite and classified by KK-theory). It is known that $1 \Rightarrow 2 \Rightarrow 3$ in full generality. However, it is not known if $3 \Rightarrow 1$ (but it is true under restriction on the trace space).

10 Dynamical systems and Rokhlin dimension

In this section we consider dynamical systems and C*-algebras which can be associated with then in a natural way. We also introduce the notion of Rokhlin dimension and relate it to the nuclear dimension of discussed C*-algebras.

Let G be a discrete group and A denotes some C*-algebra. Action α of G on A is a map $\alpha : G \to \mathrm{Aut}(A)$ (we write $G \overset{\alpha}{\curvearrowright} A$). The pair (G, α) is so called C*-dynamical system. Observe that if $A = C_0(X)$ then $G \overset{\alpha}{\curvearrowright} A$ defines an action α of G on X.

Starting from C*-dynamical system we can construct universal crossed product $A \rtimes_\alpha G$. One can think of this as a C*-algebra generated by element $a \in A$ and unitaries u_g where $g \in G$ and $u_g a u_g^* = \alpha_g(a)$, i.e.

$$A \rtimes_\alpha G = C^*(a, u_g : u_g a u_g^* = \alpha_g(a), a \in A, g \in G).$$

Starting form concrete faithful representation $A \hookrightarrow B(H)$ we can consider also the reduced cross product $A \rtimes_{\alpha,r} G \subseteq B(\ell^2(G) \otimes H)$ (it appears that $A \rtimes_{\alpha,r} G$ does not depend on the choice of this faithful representation of A). $A \rtimes_{\alpha,r} G$ is generated by elements $\lambda_g \in B(\ell^2(G) \otimes H)$ and $\pi(a) \in B(\ell^2(G) \otimes H)$ such that

$$\pi(a)(e_g \otimes \xi) = e_g \otimes \alpha_{g^{-1}}(a)\xi,$$

$$\lambda_h(e_g \otimes \xi) = e_{hg} \otimes \xi,$$

where $g \in G$ and $a \in A$. In the case of G finite it is convenient to use matrix units $e_{g,h}$. In that case

$$\pi(a) = \sum_{g \in G} e_{g,g} \otimes \alpha_{g^{-1}}(a),$$

$$\pi(a)\lambda_h = \sum_{g \in G} e_{g,h^{-1}g} \otimes \alpha_{g^{-1}}(a).$$

Therefore, we have a natural way to consider $A \rtimes_{\alpha,r} G$ as a subalgebra of $M_{|G|}(A)$.

Fact 10.1. *If G i amenable then $A \rtimes_\alpha G \cong A \rtimes_{\alpha,r} G$ (in this case we will omit subscript r).*

We want to consider a natural question: when $\dim_{\mathrm{nuc}}(A \rtimes_\alpha G) < \infty$? We wish to find an estimate which involves nuclear dimension of A, dimension of G and action α (in a certain sense).

Definition 10.2 (Oceneau, Herman). Let G be a finite group and $G \overset{\alpha}{\curvearrowright} A$ with A being a unital C*-algebra. A has a Rokhlin property if the following is true: for any finite subset $\mathcal{F} \Subset A$ and any $\epsilon > 0$ there exists a sequence of projection $(p_g)_{g \in G} \subseteq A$ such that

1. $\sum_{g \in G} p_g = \mathbb{1}$,

2. $\|\alpha_g(p_h) - p_{gh}\| < \epsilon$,

3. $\|[p_g, a]\| < \epsilon$ for all $g \in G$ and $a \in \mathcal{F}$.

One can generalized this notion to the special case $G = \mathbb{Z}$.

Definition 10.3 (Oceneau, Herman). Let $\mathbb{Z} \overset{\alpha}{\curvearrowright} A$ with A being unital C*-algebra. A has a (cyclic) Rokhlin property if the following is true: for any finite subset $\mathcal{F} \Subset A$, any $\epsilon > 0$ an any $n \in \mathbb{N}$ there exist projections $p_0, p_1, \ldots, p_{n-1} \in A$ such that

1. $p_0 + p_1 + \ldots + p_{n-1} = \mathbb{1}$,

2. $\|\alpha_1(p_i) - p_{i+1}\| < \epsilon$ (with i mod n),

3. $\|[p_i, a]\| < \epsilon$ for all $i = 0, 1, \ldots, n-1$ and $a \in \mathcal{F}$.

Let us present the classical motivation which is behind that definitions.

Theorem 10.4 (Rokhlin lemma)**.** *Let $T : X \to X$ be an aperiodic (such that periodic points have measure zero) measure transformation of Lebesgue measure space (X, μ). Then for any $\epsilon > 0$ and any $n \in \mathbb{N}$ there exists measurable subset $E \subseteq X$ such that*

1. *$E, TE, T^2E, \ldots T^{n-1}E$ are pairwise disjoint sets,*

2. *$\mu(E \cup TE \cup T^2E \cup \ldots \cup T^{n-1}E) > 1 - \epsilon$.*

Condition 2 from the previous theorem can be stated as a demanding that characteristic functions $\chi_E, \chi_{TE}, \chi_{T^2E}, \ldots, \chi_{T^{n-1}E}$ give approximate partition of unit.

Proposition 10.5. *Let G be finite group and $G \overset{\alpha}{\curvearrowright} A$ has Rokhlin property. Then* $\dim_{\mathrm{nuc}}(A \rtimes_\alpha G) \leqslant \dim_{\mathrm{nuc}}(A)$.

sketch of a proof. Let $\dim_{\mathrm{nuc}}(A) = d$ and let us fixed $\mathcal{F}$, ϵ and projections $(p_g)_{g \in G} \subseteq A$. $\mathcal{F} \times G$ is a finite subset in $A \rtimes_\alpha G$. We have the following diagram

$$\begin{array}{ccccc} A \rtimes_\alpha G & & & & A \rtimes_\alpha G \\ \cap\!\!| & & & & \rho\uparrow \\ M_{|G|}(A) & & & & M_{|G|}(A) \\ & \overset{\psi}{\searrow} & & \overset{\varphi}{\nearrow} & \\ & & F^{(0)} \oplus \ldots \oplus F^{(d)} & & \end{array}$$

where ρ is given by

$$\rho(e_{g,h} \otimes a) = p_g u_g a u_h^* p_h$$

with $e_{g,h}$ denoting basis in $M_{|G|}(A)$. Observe that $\rho(au_g) \approx au_g$ for any $a \in \mathcal{F}$ (exercise) and ρ is approximately homomorphism (considered as a map to $M_{|G|}(\mathcal{F})$), so $\rho\varphi_i$ are approximately order zero maps and can be perturbed into order zero maps. □

Proposition 10.6. *Let $\mathbb{Z} \overset{\alpha}{\curvearrowright} A$ has Rokhlin property. Then* $\dim^+_{\mathrm{nuc}}(A \rtimes_\alpha \mathbb{Z}) \leqslant 2 \dim^+_{\mathrm{nuc}}(A)$.

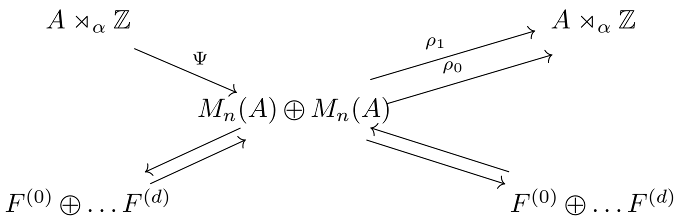
sketch of a proof. Let us fix $\mathcal{F}, \epsilon, n$ and set of projections $p_0, \ldots, p_{n-1}$. Recall that $A \rtimes_\alpha \mathbb{Z} \subseteq B(\ell^2(\mathbb{Z}) \otimes H)$. Define projections P_n and P'_n as projections on subspaces H_n and H'_n given by

$$\langle e_0, \ldots, e_{n-1} \rangle \otimes H$$

and respectively

$$\left\langle e_{\lfloor (\frac{n}{2}) \rfloor}, \ldots, e_{\lfloor (\frac{3n}{2}) \rfloor} \right\rangle \otimes H.$$

where e_i denote basis vectors in $\ell^2(\mathbb{Z})$. We have the following diagram



where $\rho_0, \rho_1 : M_n(A) \to A \rtimes_\alpha \mathbb{Z}$ are given by

$$\rho_0(e_{ij} \otimes a) = p_i u^i a u^{-j} p_j,$$

$$\rho_1(e_{ij} \otimes a) = p_i u^i a u^{-j} p_j$$

and Ψ is cut down map defined by projections P_n, P_n' rescaled by matrix D defined in the proof of theorem 8.9, i.e. $\Psi = D(P_n \times P_n)D \oplus D(P_n' \times P_n')D$.

Since ρ_0, ρ_1 are approximately multiplicative after small perturbation all $\rho_0\varphi_i$ and $\rho_1\varphi_i$ are desired order zero maps. □

Definition 10.7. (Rokhlin dimension) Let be finite group and $G \overset{\alpha}{\curvearrowright} A$ (with A being unital). Then $\dim_{\mathrm{Rok}}(\alpha) \leqslant d$ if and only if for any finite subset $\mathcal{F} \subseteq A$ and any $\epsilon > 0$ there exists a family of elements $f_g^{(l)} \in A_+, g \in G, l = 0, \ldots, d$ such that

1. $\left\| \sum_{g,l} f_g^{(l)} - \mathbb{1} \right\| < \epsilon$,
2. $\left\| \alpha_g(f_h^{(l)}) - f_g^{(l)} h \right\| < \epsilon$ for any given l,
3. $\left\| [a, f_g^{(l)}] \right\| < \epsilon$ for any given l and $a \in \mathcal{F}$,
4. $\left\| f_g^{(l)} f_h^{(l)} \right\| < \epsilon$ for any given l and $g \neq h$.

Remark 10.8. If $\dim_{\mathrm{Rok}}(\alpha) = 0$, then we have Rokhlin property ($f_g^{(0)}$ can be slightly perturbed in order to give desired set of projections).

Definition 10.9. (Rokhlin dimension) Let $\mathbb{Z} \overset{\alpha}{\curvearrowright} A$ (with A being unital). Then $\dim_{\mathrm{Rok}}(\alpha) \leqslant d$ if and only if for any finite subset $\mathcal{F} \subseteq A$, $\epsilon > 0$ and any $n \in \mathbb{N}$ there exists a family of elements $f_i^{(l)} \in A_+, i = 0, \ldots n-1, l = 0, \ldots, d$ such that

1. $\left\| \sum_{i,l} f_i^{(l)} - \mathbb{1} \right\| < \epsilon$,
2. $\left\| \alpha_1(f_i^{(l)}) - f_{i+1}^{(l)} \right\| < \epsilon$ for any given l (i mod n),
3. $\left\| [a, f_i^{(l)}] \right\| < \epsilon$ for any given l and $a \in \mathcal{F}$,
4. $\left\| f_i^{(l)} f_j^{(l)} \right\| < \epsilon$ for any given l and $i \neq j$.

Proposition 10.10. *Let G be finite group and $G \overset{\alpha}{\curvearrowright} A$ with* $\dim_{\mathrm{Rok}}(\alpha) \leqslant d$. *Then* $\dim^+_{\mathrm{nuc}}(A \rtimes_\alpha G) \leqslant \dim^+_{\mathrm{nuc}}(A)(d+1)$.

sketch of a proof. Similarly to the proof of theorem 10.5 we have the following diagram

$$
\begin{array}{ccccc}
A \rtimes_\alpha G & & & & A \rtimes_\alpha G \\
\cap\!\!| & & & & \uparrow{\scriptstyle \rho_0,\dots,\rho_d} \\
M_{|G|}(A) & & & & \bigoplus_0^d M_{|G|}(A) \\
& \overset{\psi}{\searrow} & & \overset{\varphi}{\nearrow} & \\
& & F^{(0)} \oplus \ldots \oplus F^{(d)} & &
\end{array}
$$

with $\rho_l : M_{|G|}(A) \to A \rtimes_\alpha G$ given by

$$\rho_l(e_{g,h} \otimes a) = (f_g^{(l)})^{\frac{1}{2}} u_g a u_h^* (f_g^{(l)})^{\frac{1}{2}}. \tag{10.1}$$

□

Proposition 10.11. *Let* $\mathbb{Z} \overset{\alpha}{\curvearrowright} A$ *with* $\dim_{\mathrm{Rok}}(\alpha) \leqslant d$*. Then* $\dim^+_{\mathrm{nuc}}(A \rtimes_\alpha \mathbb{Z}) \leqslant 2 \dim^+_{\mathrm{nuc}}(A)(d+1)$.

sketch of a proof. Similar to the proof of proposition 10.6. □

Question: How prevalent is Rokhlin dimension?

Theorem 10.12 (Szabo for $\mathbb{Z}^n$ action, Winter, Krichberg, Zacharias)**.** *If* X *is compact metric space with free action* $\mathbb{Z} \overset{\alpha}{\curvearrowright} C(X)$ *then* $\dim_{\mathrm{Rok}}(\alpha) \leqslant 2\dim(X)+1$ *(where* $\dim(X)$ *denotes topological dimension).*

sketch of a proof. Proof is based on the following lemma (Gutman, 2012) - topological version of Rokhlin lemma: Suppose X is compact metric space with free action $\mathbb{Z} \overset{\alpha}{\curvearrowright} C(X)$. Let $\dim(X) = d < \infty$ and $n \in \mathbb{N}$, then there exists $U \subseteq X$ open set such that

1. $\overline{U}, \alpha(\overline{U}), \ldots \alpha^{n-1}(\overline{U})$ are pairwise disjoint sets,
2. $X = \bigcup_{j=0}^{2(d+1)-1} \alpha^j(\overline{U})$.

□

11 Rokhlin dimension for residually finite groups

We have already seen the importance of the concept of Rokhlin dimension. In this section we extend this notion beyond the case of $\mathbb{Z}$ and finite groups. Let us start with the following definition.

Definition 11.1. Let G be a discrete group. We say that G is residually finite if for any $x \in G \backslash \{e\}$ there exist finite group F and homomorphism $\varphi : G \to F$ such that $\varphi(x) \neq e$ (in other words $G \hookrightarrow \prod G/N$ is injective, where G/N denotes finite quotient).

Example 11.2. The following groups are residually finite:

- $\mathbb{Z}$, $\mathbb{Z}^d$,

- Heisenberg group $\left\{ M = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$,
- $\mathrm{SL}_n(\mathbb{Z})$,
- free groups,
- all group of polynomial growth.

Let G be a countable and residually finite group. Then we can find a sequence (G_n) of normal finite subgroups such that $G \hookrightarrow \prod G/G_n$ is injective ((G_n) is known as residually finite approximation of G).

Definition 11.3. Let G and (G_n) be as before with $G \overset{\alpha}{\curvearrowright} A$ and A being unital. Then $\mathrm{Rok}(\alpha) \leqslant d$ if and only if for any $\epsilon > 0$, any finite subset $\mathcal{F} \ll A$ and any $n \in \mathbb{N}$ there exists a sequence $(f_{\overline{g}}^{(l)}) \subseteq A_+, l = 0, 1, \ldots, d, \overline{g} \in G/G_n$ such that

1. $\left\| \sum_{\overline{g},l} f_{\overline{g}}^{(l)} - \mathbb{1} \right\| < \epsilon$,
2. $\left\| \alpha_{\overline{g}}(f_{\overline{h}}^{(l)}) - f_{\overline{gh}}^{(l)} \right\| < \epsilon$ for any given l,
3. $\left\| [a, f_{\overline{g}}^{(l)}] \right\| < \epsilon$ for any given l and $a \in \mathcal{F}$,
4. $\left\| f_{\overline{g}}^{(l)} f_{\overline{h}}^{(l)} \right\| < \epsilon$ for any given l and $\overline{g} \neq \overline{h}$.

Remark 11.4. When $G = \mathbb{Z}$ and $G_n = n\mathbb{Z}$ we get definition 10.9.

One can reformulate the previous definition in a more elegant way. Let A be unital and separable. Recall that

$$A_\infty = \ell^\infty(A)/c_0(A)$$

and

$$A \hookrightarrow A_\infty$$

We can consider the central sequence algebra $F(A) = A_\infty \cap A'$. If $G \overset{\alpha}{\curvearrowright} A$ then we can obtain $G \overset{\alpha_\infty}{\curvearrowright} A_\infty$ and $G \overset{\alpha_\infty}{\curvearrowright} F(A)$. Given G and (G_n) as before, one can obtain $G \overset{\sigma_n}{\curvearrowright} C(G/G_n)$ given by

$$\sigma_n(h) : e_{\overline{g}} \mapsto e_{\overline{hg}}.$$

Proposition 11.5. *Let G and (G_n) be as before. $G \overset{\alpha}{\curvearrowright} A$ satisfies $\mathrm{Rok}(\alpha) \leqslant d$ if and only if for any $n \in \mathbb{N}$ there exist G-equivariant order zero maps $\varphi_0, \ldots \varphi_d$*

$$\varphi_l : (C(G/G_n), \sigma_n) \to (F(A), \alpha_\infty)$$

such that $\varphi_0(\mathbb{1}) + \ldots + \varphi_d(\mathbb{1}) = \mathbb{1}$.

sketch of a proof. Define $\varphi_l(e_{\overline{g}}) = [(f_{\overline{g}}^{(l)}(\epsilon_n, \mathcal{F}_n))] \in \ell^\infty(A)/c_0(A)$ with $\epsilon_n \to 0$ and $\mathcal{F}_n \to A$. □

Remark 11.6. This formalism can be generalized to the setting of topological groups, e.g. $G = \mathbb{R}$ and G_n is replaced by $\mathbb{Z}x,\ x > 0$. Then $G/G_n = \mathbb{R}/\mathbb{Z}x \cong \mathbb{T}$. One can develop theory of Rokhlin dimension for compact groups (Gardella).

Definition 11.7 (Roe)**.** Let G be a residually finite group with residually finite approximation (G_n). Assume that G is finitely generated with word length metric l. The box space

$$\Box\, G = \Box_{(G_n)}\, G = \coprod_n G/G_n$$

is defined as a discrete metric space where G/G_n carries the word length metric l in the quotient and G/G_n, G/G_m are "far apart" for $n \neq m$.

Theorem 11.8 (Roe)**.** *G is amenable if and only if $\Box\, G$ has property A. G is exact if and only if G as a space has property A. Moreover,* $\operatorname{asdim}(X) < \infty$ *implies that X has property A (inverse implication is not true in full generality).*

Remark 11.9. Here property A is some sort of 2-variable amenability of coarse metric space.

Question: When do we have $\operatorname{asdim}(\Box\, G) < \infty$? In other words, does $\operatorname{asdim}(\Box\, G) < \infty$ imply that G is amenable?
Partial answer: $\operatorname{asdim}(\Box\, G) < \infty$ if G is finitely generated, nilpotent/polynomial growth (Szabo, Wu, Zacharias).

Theorem 11.10 (Szabo, Wu, Zacharias)**.** *Let G be residually finite and A be a unital C^*-algebra.*

1. *If $G \overset{\alpha}{\curvearrowright} A$ then* $\dim^+_{\mathrm{nuc}}(A \rtimes_\alpha G) \leqslant (\dim^+_{\mathrm{nuc}}(A))(\dim^+_{\mathrm{Rok}}(\alpha))(\operatorname{asdim}^+(\Box\, G))$.
2. *If X is compact metric space and $G \overset{\alpha}{\curvearrowright} C(X)$ is free and G is nilpotent with finite Hirsch length $l = l_{Hirsch}(G) < \infty$, then* $\dim^+_{\mathrm{Rok}}(\alpha) \leqslant 3^l(\dim^+(X))$ *so* $\dim^+_{\mathrm{nuc}}(C(X) \rtimes_\alpha G) \leqslant 3^l(\dim^+(X))^2(\operatorname{asdim}^+(\Box\, G))$.